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The algebra of flat currents for the string on $AdS_5 \times S^5$ in the light-cone gauge

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Abstract: We continue the program initiated in [41] and calculate the algebra of the flat currents for the string on $AdS_5 \times S^5$ background in the light-cone gauge with κ symmetry fixed. We find that the algebra has a closed form and that the non-ultralocal terms come with a weight factor $e^{\phi(\sigma)}$ that depends on the radial AdS_5 coordinate. Based on results in two-dimensional sigma models coupled to gravity via the dilaton field, this suggests that the algebra of transition matrices in the present case is likely to be unambigous.

Keywords: Integrable Field Theories, AdS-CFT and dS-CFT Correspondence, Sigma Models.

Contents

1. Introduction

Recently there has been much interest in studying the integrable structures of the $\mathcal{N} = 4$ SYM theory as well as of strings propagating on the $AdS_5 \times S^5$ background with the goal of gaining a deeper insight into the AdS/CFT correspondence [1−4]. In this context, significant progress has been made on the SYM side in determining the anomalous dimensions of conformal operators and comparing them with the energy spectrum of the string states [5−28] (for recent reviews see [29−34] and references therein). It is becoming increasingly evident from the recent developments in this direction that the Bethe ansatz provides the main scheme suitable for understanding the quantum integrability of this system. On the gauge theory side, the Bethe ansatz is known up to three-loop order in the $su(2)$ sector and more recently has been generalized to all-loop order in [35, 36]. On the string side, it was clarified in [37, 38] how classical Bethe type equations appear in the $su(2)$ sub-sector of the string. Their relation with that of the spin chain on the SYM side at one loop order is then established in the scaling limit when the effective coupling constant $\lambda/L^2 \longrightarrow 0$. Intuitively, it is reasonable to expect that in parallel to what happens in the gauge theory side, there exists some discretized version of the Bethe ansatz for the quantum string leading to correct results in the classical limit. Indeed, a major progress in this direction has been made in [39] where such a remarkable consistent discretization has been constructed. The origin of this construction, however, is not well understood, and it is not obvious how to derive such discrete Bethe equations in general, although there exist some helpful clues in the $su(2)$ sector considered in [37]. Indeed, the sigma-model string action on $AdS_1 \times S^3$ takes a particularly simple form, and when subjected to the Virasoro constraints, this model coincides with the interacting two-spin model of Faddeev-Reshetikhin which is known to be quantum integrable [40]. In the $su(2)$ sub-sector of the string, it is obvious that the Virasoro constraints play a crucial role. It is not clear, however, whether a similar simplification may arise for the string on $AdS_5 \times S^5$ as well.

It is well known that sigma models (both bosonic as well as supersymmetric) lead to a classical current algebra (Poisson bracket algebra) that involves Schwinger terms (derivatives of delta function which are also known as non-ultralocal terms). In an earlier paper [41] the presence of such terms in the classical current algebra has been explicitly verified for the string on $AdS_5 \times S^5$. Even if one sets the fermions of the theory to zero (thus temporarily avoiding the technical difficulties associated with the local κ -symmetry), the classical current algebra which has a closed form contains Schwinger (non-ultralocal) terms. It is known [42] that in the principal chiral model in flat space-time, such terms lead to difficulties (nonuniqueness) in the calculation of the algebra of the monodromy matrices which is essential for the quantization of the model and generally one needs a regularization procedure to define this algebra. One interesting (but not well-understood) method to deal with such difficulties was proposed by Faddeev and Reshetikhin (FR) [40] and involves reducing (by hand) the original theory to the one of the two interacting spins where the classical current algebra does not have non-ultralocal terms. The original classical current algebra is then argued to be recovered in a rather special limiting procedure in the quantum version of the reduced model. In the $su(2)$ sub-sector considered in [37], the Virasoro constraints effectively remove these non-ultralocal terms. One may, then, ask whether the Virasoro constraints alone are powerful enough to provide a consistent reduction of the full $AdS_5 \times S^5$ sector to a FR type model, and whether this depends in any way on the gauge-fixing scheme. The answer to this is not at all obvious because of subtleties in gauge fixing in a curved background. We would also like to point out here another relevant interesting phenomenon which occurs in the 2-dimensional sigma model coupled to gravity through the dilaton field [43]. Here, due to the fact that the spectral parameter is a local function of the world-sheet and explicitly depends on the dilaton field through $\rho(x) \sim e^{\phi}$, the non-ultralocal terms in the current algebra are manageable and the algebra of the monodromy matrices is well defined. Basically, in this case the dilaton field regularizes the difficulties arising from the non-ultralocal terms in the classical current algebra. However, because of conformal invariance, the string on $AdS_5 \times S^5$ has no coupling to the dilaton and as a result this simple feature of sigma models on a curved background cannot be directly used in the calculation of the algebra of the monodromy matrices. In this note we will explore the integrability properties of the string on $AdS_5 \times S^5$ background in the light-cone gauge and attempt to explicitly calculate the current algebra. We will show that

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in the light-cone gauge similar effective dependence on the radial coordinate of AdS_5 also arises in the algebra of currents for the string on $AdS_5 \times S^5$ which may be helpful in the calculation of the algebra of the monodromy matrices (and, therefore, in the quantization of the model).

One of the main motivations for choosing the light-cone gauge is to simplify the covariant superstring action thereby making it more amenable for algebraic calculations. The covariant Green-Schwarz formulation of the string on $AdS_5 \times S^5$ background [44], for example, is a non-linear sigma model on the supercoset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ and the action is rather formidable. For instance, because of κ symmetry as well as reparameterization invariance, the action contains fermionic terms up to order θ^{32} which makes it rather impractical. It is therefore necessary to fix these local symmetries. Indeed, as shown in [45, 46] the superstring on $AdS_5 \times S^5$ background significantly simplifies if one employs the Killing gauge for the κ-symmetry: the resulting action will be limited to order θ^4 in the fermionic terms. Past experience with strings propagating in a flat background suggests that the light-cone gauge may be the most appropriate if one has to quantize the string on $AdS_5 \times S^5$ background.[1] This can be easily seen from the point of view of the hamiltonian analysis of the covariant superstring. The fermionic constraints of the theory resulting from the κ-symmetry lead to a complicated mixture of the first and second class constraints which are impossible to disentangle covariantly. As a result, the fundamental Dirac brackets for the theory are hard to construct.[2] On the other hand, the fundamental Dirac brackets are absolutely necessary to determine the classical r-matrix. The only solution, therefore, is to fix appropriately the κ-symmetry and this can be conveniently done in the light-cone gauge.

Our main goal lies in calculating the algebra of the flat currents depending only on the physical fields (variables). It is clear from the above discussion, that in order to do this, we must fix both the bosonic reparameterization invariance (with the light-cone gauge) as well as the local κ-symmetry. The consistent light-cone formulation of the string on $AdS_5 \times S^5$ background has been discussed by Metsaev and Tseytlin in [48] and further elaborated using phase space lagrangian in [49]. We mention here only one important feature of this gauge fixing that will play a significant role in our calculations. Namely, unlike in the flat space case, it is impossible to fix simultaneously the conventional bosonic light-cone gauge $x^+ = \tau$ and $g^{\mu\nu} = \eta^{\mu\nu}, \mu, \nu = 0, 1$ consistent with the equations of motion for the string on the $AdS_5 \times S^5$ background [50, 51]. Instead one can choose $x^+ = \tau, \mathcal{P}^+ = \mathrm{const}$ in the phase space, which translates into $h^{00} = -p^+$, $h^{11} = (p^+)^{-1} e^{4\phi}$, $h^{01} = h^{10} = 0$ $\left(h^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} e^{2\phi}\right)$, in the coordinate space. As discussed in [52] the Virasoro constraints do not follow from the Lax representation of the sigma model and have to be considered separately, and this particular gauge choice corresponds to solving the Virasoro constraints explicitly. It is this feature of the light-cone gauge that leads to the appearance of the e^{ϕ} factor in the non-ultralocal terms in the current algebra.

[1] Light cone gauge is also useful in the study of the $\mathcal{N} = 4$ super Yang-Mills theory, which can be formulated on superspace in the light cone gauge. By using such a formulation, integrability of the anomalous dimension in the SL(2) sector at one and two loops has been studied for SYM theory in [26, 27, 34, 28].

[2] There has been, however, some progress in covariant quantization using the *pure spinor formalism* (for a review see [47]).

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The paper is organized as follows. In section 2 we briefly summarize the main properties of the superstring on the $AdS_5 \times S^5$ background. In section 3 we review the necessary results in the light-cone gauge that are used in our calculations. In section 4 we present the algebra of currents without the spectral parameter in the light-cone gauge and show that is has a closed form. In section 5 we present the algebra of currents with the spectral parameter in the light-cone gauge. Here, we find that, in general, the algebra is also closed. We also point out that potentially dangerous non-ultralocal terms in the algebra come with a weight factor proportional to $e^{\phi(\sigma)}$. This, therefore, is likely to lead to an unambiguous algebra for the transition matrices. In appendices A–E we present all the technical details that are not given in the main text.

2. Review of the superstring on $AdS_5 \times S^5$

We summarize here some of the basic properties of the type-IIB Green-Schwarz superstring action on the $AdS_5 \times S^5$ background [45, 46, 48, 53]. The superstring can be defined as a non-linear sigma model on the coset superspace

$$\frac{G}{H} = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1) \times \mathrm{SO}(5)} \,. \tag{2.1}$$

The classical action has the Wess-Zumino-Witten form

$$S = -\frac{1}{2} \int\limits_{\partial M^3} \mathrm{d}^2\sigma \sqrt{-g} g^{\mu\nu} \left(L_\mu^{\hat{A}} L_\nu^{\hat{A}} \right) + i \int\limits_{M^3} s^{IJ} \left(L^{\hat{A}} \wedge \overline{L}^I \hat{\gamma}^{\hat{A}} \wedge L^J \right), \tag{2.2}$$

where $g^{\mu\nu}, \mu, \nu = 0, 1$ represents the worldsheet metric, $\hat{A} = (A, A')$ with $A = (0, \ldots, 4)$ and $A' = (5, \ldots, 9)$ denotes the tangent space indices for AdS_5 and S^5 respectively, $s^{IJ} = \mathrm{diag}(1, -1), I, J = (1, 2)$; $\hat{\gamma}^A \equiv \gamma^A$, $\hat{\gamma}^{A'} \equiv i\gamma^{A'}$ ($\gamma^{(A,A')}$ represent ten dimensional Dirac matrices), $s^{IJ} = \mathrm{diag}(1, -1), I, J = (1, 2)$. We use the convention that repeated indices are summed. The supervielbeins $L^{\hat{A}}$ and L^I are defined by the left-invariant Cartan 1-forms in the $so(4,1) \oplus so(5)$ basis of $psu(2,2|4)$ (lower case letters denote Lie algebras) as follows

$$G^{-1}\mathrm{d}G = \hat{L}^A \hat{P}_A + L^{A'} P_{A'} + \frac{1}{2} \hat{L}^{AB} \hat{J}_{AB} + \frac{1}{2} L^{A'B'} J_{A'B'} + L^{\alpha\alpha' I} Q_{\alpha\alpha' I} \,, \tag{2.3}$$

where $\alpha, \alpha' = 1, 2, \ldots, 32$ $L^{\hat{A}} = \left(\hat{L}^A, L^{A'} \right)$ and with

$$L^{\hat{A}} = \mathrm{d}X^M L_M^{\hat{A}} \,, \tag{2.4}$$

and $X^M = (x, \theta^I)$ represent the bosonic and fermionic string coordinates in the target space.

By solving the Cartan-Maurer equation (zero curvature condition) one can explicitly determine

$$L^I = \left(\left(\frac{\sinh \mathcal{M}}{\mathcal{M}} \right) D\theta \right)^I ,$$

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$$L^{\hat{A}} = e^{\hat{A}}(x)\mathrm{d}x - i\bar{\theta}\gamma^{\hat{A}}\left(\left(\frac{\sinh \mathcal{M}/2}{\mathcal{M}/2}\right)^2 D\theta\right), \tag{2.5}$$

where

$$\left(\mathcal{M}^2\right)^{IJ} = \epsilon^{IK}\left(-\gamma^A\theta^K\bar{\theta}^J\gamma^A + \gamma^{A'}\theta^K\bar{\theta}^J\gamma^{A'}\right) + \frac{1}{2}\epsilon^{KJ}\left(\gamma^{AB}\theta^I\bar{\theta}^K\gamma^{AB} - \gamma^{A'B'}\theta^I\bar{\theta}^K\gamma^{A'B'}\right). \tag{2.6}$$

Denoting by $\left(e^{\hat{A}}, \omega^{\hat{A}\hat{B}}\right)$ the bosonic vielbein and the spin connection respectively, the covariant differential of the fermions is given by

$$(D\theta)^I = \left[\delta^{IJ}\left(\mathrm{d} + \frac{1}{4}\omega^{\hat{A}\hat{B}}\gamma_{\hat{A}\hat{B}}\right) - \frac{i}{2}\epsilon^{IJ}\, e^{\hat{A}}\gamma_{\hat{A}}\right]\theta^J\,. \tag{2.7}$$

The equations of motion following from action (2.2) take the forms

$$\sqrt{-g}g^{\mu\nu}\left(\nabla_\mu \hat{L}^A_\nu + \hat{L}^{AB}_\mu \hat{L}^B_\nu\right) + i\epsilon^{\mu\nu}s^{IJ}\bar{L}^I_\mu\gamma^A L^J_\nu = 0\,, \tag{2.8}$$

$$\sqrt{-g}g^{\mu\nu}\left(\nabla_\mu L^{A'}_\nu + L^{A'B'}_\mu L^{B'}_\nu\right) - \epsilon^{\mu\nu}s^{IJ}\bar{L}^I_\mu\gamma^{A'} L^J_\nu = 0\,, \tag{2.9}$$

$$\left(\gamma^A L^A_\mu + i\gamma^{A'}L^{A'}_\mu\right)\left(\sqrt{-g}g^{\mu\nu}\delta^{IJ} - \epsilon^{\mu\nu}s^{IJ}\right)L^J_\nu = 0\,, \tag{2.10}$$

with ∇_μ representing the covariant derivative on the worldsheet.

To consider the integrability properties of the sigma model we will need some of the properties of the superalgebra $psu(2,2|4)$ which we briefly review in appendix A. Let us consider the map G from the string worldsheet into the graded group PSU$(2,2|4)$. In this case, the current 1-form $J = -G^{-1}\mathrm{d}G$ belongs to the superalgebra and, therefore, can be decomposed as

$$J = -G^{-1}\mathrm{d}G = H + P + Q^1 + Q^2\,, \tag{2.11}$$

where, using the notations in appendix A, we can identify

$$H = H_0\,, \qquad Q^1 = H_1\,, \qquad P = H_2\,, \qquad Q^2 = H_3\,. \tag{2.12}$$

In terms of $L^{\hat{A}}$, $L^{\hat{A}\hat{B}}$ and L^I one can also read off the following expressions using (2.3):

$$\begin{aligned} H &= \frac{1}{2}\hat{L}^{AB}\hat{J}_{AB} + \frac{1}{2}L^{A'B'}J_{A'B'}\,,\\ P &= \hat{L}^A\hat{P}_A + L^{A'}P_{A'}\,,\\ Q^I &= L^{\alpha\alpha' I}Q_{\alpha\alpha' I}\,. \end{aligned} \tag{2.13}$$

From the definition of the current in (2.11), we see that it satisfies the zero curvature condition

$$\mathrm{d}J - J\wedge J = 0\,. \tag{2.14}$$

In terms of the components of the current (2.12), the equations of motion can be written as [7]

$$\mathrm{d}\,{}^*P = {}^*P\wedge H + H\wedge {}^*P + \frac{1}{2}\left(Q\wedge Q' + Q'\wedge Q\right),$$

$$
\begin{aligned}
0 &= P \wedge ({}^{*}Q - Q') + ({}^{*}Q - Q') \wedge P\,, \\
0 &= P \wedge (Q - {}^{*}Q') + (Q - {}^{*}Q') \wedge P\,,
\end{aligned} \tag{2.15}
$$

where * denotes the Hodge star operation and we have defined $Q \equiv Q^1 + Q^2, Q' \equiv Q^1 - Q^2$.

Classical integrability is established by writing down a one parameter family of currents $\hat{J}(t) \equiv -\hat{G}^{-1}(t)\mathrm{d}\hat{G}(t)$, where t is a constant spectral parameter, satisfying the flatness condition [7]. We will use here the convenient form of the one parameter family of currents presented in [41] (here we are suppressing the dependence on the worldsheet coordinates of all variables)

$$
\hat{J}(t) = H + \frac{1+t^2}{1-t^2}\,P + \frac{2t}{1-t^2}\,{}^{*}P + \sqrt{\frac{1}{1-t^2}}\,Q + \sqrt{\frac{t^2}{1-t^2}}\,Q'\,, \tag{2.16}
$$

such that $\hat{J}(t=0) = J$ of (2.11). It is easy to check that the vanishing curvature condition for this new current

$$
\mathrm{d}\hat{J} - \hat{J} \wedge \hat{J} = 0\,, \tag{2.17}
$$

leads to all the equations of motion (2.15) as well as the zero curvature condition (2.14).

The sigma model action which leads to the equations of motion (2.15) has the form

$$
S = \frac{1}{2}\int \mathrm{str}\left(P \wedge {}^{*}P - Q^1 \wedge Q^2\right), \tag{2.18}
$$

The constraint analysis for this theory can be carried out in a straight forward manner and the resulting hamiltonian including the primary constraints has the form [41]

$$
\mathcal{H}_T = \mathrm{str}\left(\frac{1}{2}\left(P_0^2 + P_1^2\right) + \lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3\right), \tag{2.19}
$$

where λ_1, λ_2 and λ_3 denote the Lagrange multipliers corresponding to the three primary constraints

$$
\varphi_1 = -\partial_1\mathcal{P}_H + [H_1, \mathcal{P}_H] + [P_1, \mathcal{P}] + [Q_1^1, \mathcal{P}_{Q^2}] + [Q_1^2, \mathcal{P}_{Q^1}] \approx 0\,, \tag{2.20}
$$

$$
\varphi_2 = -\frac{1}{2}Q_1^1 - \partial_1\mathcal{P}_{Q^1} + [H_1, \mathcal{P}_{Q^1}] + [P_1, \mathcal{P}_{Q^2}] + [Q_1^1, \mathcal{P}_H] + [Q_1^2, \mathcal{P}_P] \approx 0\,, \tag{2.21}
$$

$$
\varphi_3 = \frac{1}{2}Q_1^2 - \partial_1\mathcal{P}_{Q^2} + [H_1, \mathcal{P}_{Q^2}] + [P_1, \mathcal{P}_{Q^1}] + [Q_1^1, \mathcal{P}_P] + [Q_1^2, \mathcal{P}_H] \approx 0\,. \tag{2.22}
$$

Here $\mathcal{P}$ denotes the canonical momenta corresponding to the respective variables. The first constraint φ_1 is easily checked to be stationary under the evolution of the system while requiring the constraints φ_2, φ_3 to be stationary determines two of the Lagrange multipliers to correspond to

$$
\lambda_2 = -Q_1^2\,, \qquad \lambda_3 = Q_1^1\,. \tag{2.23}
$$

In the conformal gauge adopted in [41] the three primary constraints $(\varphi_1, \varphi_2, \varphi_3)$ should be supplemented with the standard Virasoro constraints, which in the present notation take the forms

$$
\varphi_4 = \frac{1}{2}\,\mathrm{str}(P_0^2 + P_1^2) \approx 0\,, \tag{2.24}
$$

$$\varphi_5 = \mathrm{str}(P_0 P_1) \approx 0\,, \tag{2.25}$$

As explained in [41] it is, however, a linear combination of these constraints, namely,

$$\begin{aligned}\bar{\varphi}_4 &= \varphi_4 + \mathrm{str}\left(\lambda_2\varphi_2 + \lambda_3\varphi_3\right) \approx 0\,,\\ \bar{\varphi}_5 &= \varphi_5 - \mathrm{str}\left(\lambda_2\varphi_2 - \lambda_3\varphi_3\right) \approx 0\,,\end{aligned} \tag{2.26}$$

which can be easily checked to correspond to first class constraints and generate the transformations associated with the reparameterization invariance. The two fermionic constraints (φ_2, φ_3) are reducible with half of them being first class constraints responsible for the κ-symmetry. It is however not possible to split (φ_2, φ_3) into the first and second classes in a covariant manner. As a result, one cannot carry out the standard procedure for determining the fundamental Dirac brackets of the theory. Therefore, we are forced to give up covariant quantization and consider instead a gauge fixed theory. The most convenient and simple choice from our point of view seems to be the light-cone gauge fixed action discussed by [48] and later presented in the phase-space lagrangian formalism in [49]. In the next section we give only a short description of the key features (relevant for our work) of the light-cone gauge fixed string on $AdS_5 \times S^5$, referring the reader to the original papers [48] and [49] for details.

3. Light-cone gauge fixed string action on $AdS_5 \times S^5$

The first step in fixing the light-cone gauge is to re-write the Cartan 1-forms, given in (2.4) in the $so(4,1) \oplus so(5)$ basis, in the $so(3,1) \oplus su(4)$ basis. The explicit transformation is described in detail in [48]. We restrict here our discussion to the bosonic case for simplicity. In this basis, the bosonic part of the Cartan 1-form takes the form

$$\left(G^{-1}\mathrm{d}G\right)_{\mathrm{bosonic}} = L_P^a P_a + L_K^a K_a + L_D D + \frac{1}{2} L^{ab} J_{ab} + L_j^i J_i^j\,, \tag{3.1}$$

where the index A splits into $(a, 4)$ and the new generators correspond to those for translations, conformal boosts, dilatation

$$P^a = \hat{P}^a + \hat{J}^{4a}\,, \qquad K^a = \frac{1}{2}\left(-\hat{P}^a + \hat{J}^{4a}\right), \qquad D = -\hat{P}^4\,, \qquad J^{ab} = \hat{J}^{ab}\,, \tag{3.2}$$

and the $su(4)$ generators

$$J_j^i = -\frac{i}{2}\left(\gamma^{A'}\right)^i_j P^{A'} + \frac{1}{4}\left(\gamma^{A'B'}\right)^i_j J^{A'B'}\,. \tag{3.3}$$

In appendix B we list some relations between the old and the new basis. These bosonic generators form the $so(4,2) \oplus so(6)$ subalgebra of $psu(2,2|4)$. In the light-cone coordinates

$$\begin{aligned} x^a &= \left(x^+, x^-, x, \bar{x}\right),\\ x^\pm &\equiv \frac{1}{\sqrt{2}}\left(x^3 \pm x^0\right),\\ (x, \bar{x}) &\equiv \frac{1}{\sqrt{2}}\left(x^1 \pm i x^2\right), \end{aligned} \tag{3.4}$$

the above generators $\left(P^a, J^{ab}, K^a\right)$ split as

$$\left(P^\pm, P \equiv P^x, \bar{P} \equiv P^{\bar{x}}, J^{\pm x}, J^{\pm\bar{x}}, J^{+-}, J^{x\bar{x}}, K \equiv K^x, K^\pm, \bar{K} \equiv K^{\bar{x}}\right). \tag{3.5}$$

Since the metric in the light-cone coordinates is not diagonal, the scalar product takes the form

$$A^a B_a = A^+ B^- + A^- B^+ + A^x B^{\bar{x}} + A^{\bar{x}} B^x . \tag{3.6}$$

The fermionic generators in these coordinates split into two sets of supercharges — regular and conformal

$$Q^{\pm i}, Q_i^\pm, S^{\pm i}, S_i^\pm . \tag{3.7}$$

Finally, to fix the $\kappa-$symmetry one chooses a specific representative of the supercoset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$ parametrized by the coordinates $\left(x^a, y^i_j, \phi, \theta^{\pm i}, \theta^\pm_i, \eta^{\pm i}, \eta^\pm_i\right)$ in the following form:

$$G = G_{x,\theta} G_\eta G_y G_\phi , \tag{3.8}$$

where

$$\begin{aligned}
G_{x,\theta} &= \exp\left(x^a P_a + \theta^{-i} Q_i^+ + \theta^{+i} Q_i^- + \theta_i^- Q^{+i} + \theta_i^+ Q^{-i}\right), \\
G_\eta &= \exp\left(\eta^{-i} S_i^+ + \eta^{+i} S_i^- + \eta_i^- S^{+i} + \eta_i^+ S^{-i}\right), \\
G_y &= \exp\left(y^i_j J^j_i\right), \\
G_\phi &= \exp\left(\phi D\right),
\end{aligned} \tag{3.9}$$

where the S^5 coordinates $y^{A'}$ appear through $y^i_j \equiv \frac{i}{2}\left(\gamma^{A'}\right)^i_j y^{A'}$. Analogous to the flat space case, the κ symmetry is then fixed by setting the positive components of the spinors to zero

$$\theta^{+i} = \theta_i^+ = \eta^{+i} = \eta_i^+ = 0 . \tag{3.10}$$

The coset space representative (3.9) will, as a result, take a simpler form, and the corresponding Cartan 1-forms can be written down in the explicit form (see also the next section). The Cartan 1-forms associated with the bosonic sector are given by [48]:

$$\begin{aligned}
L_P^+ &= e^\phi \mathrm{d}x^+ , \qquad L_P^- = e^\phi \left(\mathrm{d}x^- - \frac{i}{2}\widetilde{\theta}^i \mathrm{d}\widetilde{\theta}_i - \frac{i}{2}\widetilde{\theta}_i \mathrm{d}\widetilde{\theta}^i\right), \\
L_P^x &= e^\phi \mathrm{d}x , \qquad L_P^{\bar{x}} = e^\phi \mathrm{d}\bar{x} , \qquad L_D = \mathrm{d}\phi , \\
L^i_j &= \left(\mathrm{d}U U^{-1}\right)^i_j + i\left(\widetilde{\eta}^i \widetilde{\eta}_j - \frac{1}{4}\widetilde{\eta}^2 \delta^i_j\right)\mathrm{d}x^+ ,
\end{aligned} \tag{3.11}$$

while the fermionic ones have the following form:

$$\begin{aligned}
L_K^- &= e^{-\phi}\left(\frac{1}{4}\left(\widetilde{\eta}^2\right)^2 \mathrm{d}x^+ + \frac{i}{2}\widetilde{\eta}^i \mathrm{d}\widetilde{\eta}_i + \frac{i}{2}\widetilde{\eta}_i \mathrm{d}\widetilde{\eta}^i\right), \\
L_Q^{-i} &= e^{\frac{\phi}{2}}\left(\mathrm{d}\widetilde{\theta}^i + i\widetilde{\eta}^i \mathrm{d}x\right), \qquad L_{Q_i}^- = e^{\frac{\phi}{2}}\left(\mathrm{d}\widetilde{\theta}_i - i\widetilde{\eta}_i \mathrm{d}\bar{x}\right), \\
L_Q^{+i} &= -ie^{\frac{\phi}{2}}\widetilde{\eta}^i \mathrm{d}x^+ , \qquad L_{Q_i}^+ = ie^{\frac{\phi}{2}}\widetilde{\eta}_i \mathrm{d}x^+ ,
\end{aligned}$$

$$L_S^{-i} = e^{-\frac{\phi}{2}}\left(\mathrm{d}\widetilde{\eta}^i + \frac{i}{2}\widetilde{\eta}^2\widetilde{\eta}^i\mathrm{d}x^+\right), \qquad L^-_{S_i} = e^{-\frac{\phi}{2}}\left(\mathrm{d}\widetilde{\eta}_i - \frac{i}{2}\widetilde{\eta}^2\widetilde{\eta}_i\mathrm{d}x^+\right), \tag{3.12}$$

where for simplicity we have defined

$$\begin{aligned} &\theta^i \equiv \theta^{-i}, \qquad \eta^i \equiv \eta^{-i}, \qquad \theta_i \equiv \theta^-_i, \qquad \eta_i \equiv \eta^-_i, \\ &\widetilde{\theta}^i \equiv U^i_j\theta^j, \qquad \widetilde{\theta}_i \equiv \theta_j(U^{-1})^j_i. \end{aligned} \tag{3.13}$$

The matrix U^i_j has the following explicit dependence on coordinates u^M $(M = 1\ldots 6)$ parameterizing[3] the sphere S^5

$$\begin{aligned} U^i_j &= \frac{\rho^{6ik}\left(\rho^6_{kj} + \rho^M_{kj}u^M\right)}{\sqrt{2}\sqrt{1+u^6}}, \\ \left(U^{-1}\right)^i_j &= \frac{\left(\rho^{6ik} + \rho^{Mik}u^M\right)\rho^6_{kj}}{\sqrt{2}\sqrt{1+u^6}}. \end{aligned} \tag{3.14}$$

Here ρ^M_{kj} are matrices satisfying

$$\rho^M\overline{\rho}^N + \rho^N\overline{\rho}^M = 2\delta^{MN}, \tag{3.15}$$

$$\rho^M = \left(\rho^M\right)^{ij}, \qquad \overline{\rho}^M = \left(\rho^M\right)_{ij},$$

$$\hat{\rho}^{MN} = \frac{1}{2}\left(\overline{\rho}^M\rho^N - \overline{\rho}^N\rho^M\right), \tag{3.16}$$

$$\rho^{MN} = \frac{1}{2}\left(\rho^M\overline{\rho}^N - \rho^N\overline{\rho}^M\right). \tag{3.17}$$

The gauge-fixed lagrangian density, in these coordinates and notations, takes the form

$$\begin{aligned} \mathcal{L} = &-\sqrt{-g}g^{\mu\nu}\left(e^{2\phi}\left(\partial_\mu x^+\partial_\nu x^- + \partial_\mu x\partial_\nu\bar{x}\right) + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}D_\mu u^M D_\nu u^M\right) - \\ &-\frac{i}{2}\sqrt{-g}g^{\mu\nu}e^{2\phi}\partial_\mu x^+\left(\theta^i\partial_\nu\theta_i + \eta^i\partial_\nu\eta_i + \theta_i\partial_\nu\theta^i + \eta_i\partial_\nu\eta^i + ie^{2\phi}\partial_\nu x^+\left(\eta^2\right)^2\right) + \\ &+\epsilon^{\mu\nu}e^{2\phi}\partial_\mu x^+\eta^i\left(\rho^M\right)_{ij}u^M\left(\partial_\nu\theta^j - i\sqrt{2}e^\phi\eta^j\partial_\nu x\right) + \text{h.c.}, \end{aligned} \tag{3.18}$$

where

$$D_\mu u^M \equiv \partial_\mu u^M + ie^{2\phi}\eta_i\left(\rho^{MN}\right)^i_j\eta^j u^N\partial_\mu x^+. \tag{3.19}$$

The next step is to derive the set of constraints as well as the hamiltonian. This analysis is best carried out in the (mixed) phase-space lagrangian formalism [49]. The advantage of this method is that it allows us to choose the conventional bosonic gauge

$$x^+ = \tau, \qquad \mathcal{P}^+ = \text{const.}, \tag{3.20}$$

where $\mathcal{P}^+$ is the conjugate momentum for x^-. In the flat space this translates into the usual light-cone gauge fixing in the coordinate space which is not generally consistent in a

[3] The metric on $AdS_5\times S^5$ is chosen to be of the form $\mathrm{d}s^2 = e^{2\phi}\mathrm{d}x^A\mathrm{d}x^A + (\mathrm{d}\phi)^2 + \mathrm{d}u^M\mathrm{d}u^M$, $\quad u^M u^M = 1$.

curved background [50, 51]. For the present case of the $AdS_5 \times S^5$ background, imposing the condition (3.20) in the phase space is equivalent in the coordinate space to identifying

$$h^{00} = -p^+ \,, \qquad h^{11} = \left(p^+\right)^{-1} e^{4\phi} \,, \qquad h^{01} = h^{10} = 0 \,, \tag{3.21}$$

where

$$h^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} e^{2\phi} \,. \tag{3.22}$$

Another advantage of the phase-space lagrangian is that one can easily perform explicit integration over some of the degrees of freedom, avoiding subtleties [51] associated with the ghost fields.

Even after the reparametrization and κ-symmetries are fixed, there remain some residual fermionic and bosonic constraints (e.g. $u^M u^M = 1$) and it is necessary to evaluate the Dirac brackets taking into account these and the corresponding brackets have already been obtained in [49]. The hamiltonian density has the following form:

$$\begin{aligned} \mathcal{H} = \frac{1}{2p^+} \Big[& 2\mathcal{P}\bar{\mathcal{P}} + 2e^{4\phi} x' \bar{x}' + e^{2\phi} \Big(\mathcal{P}_\phi^2 + (\phi')^2 + \mathcal{P}^M \mathcal{P}^M + u'^{\,M} u'^{\,M} + \\ & + p^{+2} (\eta^2)^2 + 4p^+ \eta_i l^i_j \eta^j \Big) \Big] - \\ & - e^{2\phi} \eta^i y_{ij} \left(\theta'^{\,j} - i\sqrt{2} e^{\phi} \eta^j x' \right) - e^{2\phi} \eta_i y^{ij} \left(\theta'_j + i\sqrt{2} e^{\phi} \eta_j \bar{x}' \right) , \end{aligned} \tag{3.23}$$

where a "prime" denotes a derivative with respect to σ and

$$\begin{aligned} y_{ij} &\equiv \left(\rho^M\right)_{ij} u^M \,, \\ l^i_j &\equiv \frac{i}{2} \left(\rho^{MN}\right)^i_j u^M \mathcal{P}^N \,. \end{aligned} \tag{3.24}$$

The fundamental Dirac brackets can be determined to be

$$\begin{aligned} \{\mathcal{P}(\sigma), \bar{x}(\sigma')\}_D &= \delta(\sigma - \sigma') = \{\bar{\mathcal{P}}(\sigma), x(\sigma')\}_D = \{\mathcal{P}_\phi(\sigma), \phi(\sigma')\}_D \,, \\ \{\mathcal{P}^M(\sigma), u^N(\sigma')\}_D &= \left(\delta^{MN} - u^M u^N\right) \delta(\sigma - \sigma') \,, \\ \{\mathcal{P}^M(\sigma), \mathcal{P}^N(\sigma')\}_D &= \left(u^M \mathcal{P}^N - u^N \mathcal{P}^M\right) \delta(\sigma - \sigma') \,, \\ \{\theta_i(\sigma), \theta^j(\sigma')\}_D &= \frac{i}{p^+} \delta^j_i \delta(\sigma - \sigma') \,, \\ \{x_0^-, \theta_i(\sigma)\}_D &= \frac{1}{2p^+} \theta_i(\sigma) \,, \qquad \{x_0^-, \theta^i(\sigma)\}_D = \frac{1}{2p^+} \theta^i(\sigma) \,. \end{aligned} \tag{3.25}$$

Here x_0^- is the zero mode of the x^-coordinate. The other brackets can be obtained from (3.25) by replacing $\left(\theta_i \to \eta_i; \theta^i \to \eta^i\right)$. We note here that the constraint $u^M u^M = 1$ leads to the secondary constraint $\mathcal{P}^M u^M = 0$ (both are second class constraints) which play a crucial role in studying the integrability properties of the model. Some related interesting and explicit calculations have recently been presented in [52].

4. Algebra of currents without the spectral parameter

In this section we would like to investigate the classical algebra of the currents with the Virasoro constraints (reflecting reparameterization invariance) and the κ-symmetry fixed.

It is not *a priori* obvious what would result in the light-cone gauge and such an algebra is desirable if one is to obtain the classical r-matrix. In [41] it was shown that (at least in the bosonic sector, when one ignores the difficulties associated with the κ-symmetry) the algebra is closed. However, the Virasoro constraints were not completely fixed there and the algebra, although closed, contained undesirable Schwinger terms. As pointed out in the introduction, the presence of such terms in the algebra of the flat currents leads to ambiguities in the computation of brackets between the transition matrices. This, in turn, prevents one from determining the classical r-matrix of the theory. Therefore, one needs to regularize such terms. There already exist several regularization schemes [54, 55, 40]. The most intriguing (and interesting) possibility has been outlined in the original work by Faddeev and Reshetikhin [40] where one introduces a reduced model "by hand", equivalent to that of the two interacting spins, and only in the end a relation between the quantum version of the reduced model and the original classical system (with the use of the Schwinger theorem) is argued in a limiting manner. This reduced system is still an interesting model to study on its own, and may turn out to be quite important after one fully understands the limiting procedure. It is quite remarkable that the FR model has resurfaced in the context of the AdS string [37], namely, the string in $AdS_1 \times S^3$ background can be written exactly as the FR model of the two interacting spins. The key point of this construction is to recognize that the Virasoro constraints in this model coincide exactly with the ones introduced by Faddeev and Reshetikhin to reduce the original system (of the principal SU(2) chiral model). In other words, there is no need to impose any condition "by hand" since the Virasoro conditions do exactly that. Motivated by these observations, it is important to analyze the effects of the Virasoro constraints when the string moves on the $AdS_5 \times S^5$ background. The ultimate goal is to elucidate whether fixing completely the reparametrization invariance will lead to some sort of well-defined spin system as it happens in the $AdS_1 \times S^3$ background.

Before we proceed, let us note that for simplicity we will employ an index free tensor notation following [56] defined as

$$\overset{1}{A} = A \otimes I\,, \qquad \overset{2}{B} = I \otimes B\,, \tag{4.1}$$

where

$$(A \otimes B)_{ij,km} = A_{ik} B_{jm}\,. \tag{4.2}$$

In these notations, our goal is to calculate $\{ \overset{(1)}{J}_1(t_{1,}\sigma), \overset{(2)}{J}_1(t_{2,}\sigma')\}_D$, where t_i is the spectral parameter, after the reparametrization and the κ-symmetry have been fixed as described in the previous section. To avoid technical complications and to illustrate the essential points, we restrict ourselves in this paper to the bosonic case and will present the full supersymmetric case in a separate publication. Let us collect the set of the light-cone generators as follows:

$$T_a = -\left(P^-, P^+, P^{\overline{x}}, P^x, D, J^i_j\right), \qquad a = 1, 2, \ldots, 6\,. \tag{4.3}$$

Then, one can write the components of the current in the following form:

$$J_0 \equiv A^a T_a\,, \tag{4.4}$$
$$J_1 \equiv B^a T_a\,. \tag{4.5}$$

The explicit form of the coefficient functions A^a and B^a can be determined using the Cartan 1-forms in the light-cone fixed gauge (see previous section). We collect below the forms of A^a and B^a which can be written as functions of the coordinates and momenta only:

$$\begin{aligned}
A^1 &= e^{\phi}\,,\\
A^2 &= -\frac{e^{\phi}}{2\,(p^+)^2}\left[2\mathcal{P}\overline{\mathcal{P}} + 2e^{4\phi}x'\overline{x}' + e^{2\phi}\left(\mathcal{P}_\phi^2 + (\phi')^2 + \mathcal{P}_M^2 + u'^{\,M}u'^{\,M}\right)\right],\\
A^3 &= \frac{e^{\phi}}{p^+}\mathcal{P}\,, \qquad A^4 = \frac{e^{\phi}}{p^+}\overline{\mathcal{P}}\,, \qquad A^5 = \frac{e^{2\phi}}{p^+}\mathcal{P}_\phi\,,\\
A^6 &= \frac{e^{2\phi}}{2\,(1+u^6)\,p^+}\left[\rho^6\widehat{\rho}^{MN}\overline{\rho}^6\mathcal{P}^M u^N - \rho^A\overline{\rho}^6\mathcal{P}^A\right],
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
B^1 &= 0\,,\\
B^2 &= -\frac{e^{\phi}}{p^+}\left[\mathcal{P}\overline{x}' + \overline{\mathcal{P}}x' + \mathcal{P}_\phi\phi' + \mathcal{P}_M u'^{\,M}\right],\\
B^3 &= e^{\phi}x'\,, \qquad B^4 = e^{\phi}\overline{x}'\,, \qquad B^5 = \phi'\,,\\
B^6 &= \frac{1}{2\,(1+u^6)}\left[\rho^6\widehat{\rho}^{MN}\overline{\rho}^6 u'^{\,M}u^N - \rho^A\overline{\rho}^6 u'^{\,A}\right],
\end{aligned} \tag{4.7}$$

where a "prime" denotes differentiation with respect to σ.

Using various relations listed in appendix B one finds that the flat current (2.16) $\hat{J}_1(\sigma,t)$ can be written as

$$\hat{J}_1(t) = H_1 + \frac{1+t^2}{1-t^2}P_1 + \frac{2t}{1-t^2}{}^*P_1\,, \tag{4.8}$$

where[4]

$$\begin{aligned}
H_1 &= B^2\left(\frac{1}{2}P^+ + K^+\right) + B^3\left(\frac{1}{2}\bar{P} + \bar{K}\right) + B^4\left(\frac{1}{2}P + K\right) + B^{6i}{}_j\frac{1}{2}(J^j{}_i + \rho^{6jk}\rho^6_{il}J^l{}_k)\,,\\
P_1 &= B^2\left(\frac{1}{2}P^+ - K^+\right) + B^3\left(\frac{1}{2}\bar{P} - \bar{K}\right) + B^4\left(\frac{1}{2}P - K\right) +\\
&\quad + B^5 D + B^{6i}{}_j\frac{1}{2}(J^j{}_i - \rho^{6jk}\rho^6_{il}J^l{}_k)\,,\\
{}^*P_1 &= -p^+e^{-2\phi}\left(A^1\left(\frac{1}{2}P^- - K^-\right) + A^2\left(\frac{1}{2}P^+ - K^+\right) + A^3\left(\frac{1}{2}\bar{P} - \bar{K}\right) +\right.\\
&\quad \left. + A^4\left(\frac{1}{2}P - K\right) + A^5 D + A^{6i}{}_j\frac{1}{2}(J^j{}_i - \rho^{6jk}\rho^6_{il}J^l{}_k)\right).
\end{aligned} \tag{4.9}$$

[4]Note that the Hodge dual is defined as ${}^*P_1 = -p^+e^{-2\phi}P_0$.

We can already see that the calculation of the algebra of currents (with the spectral parameter) is rather complicated and one requires the knowledge of the brackets (without the spectral parameter)

$$\begin{aligned}
\{\overset{(1)}{J}_0(\sigma), \overset{(2)}{J}_0(\sigma')\}_D &= \{A^a, A^b\}_D T_a \otimes T_b\,, \\
\{\overset{(1)}{J}_0(\sigma), \overset{(2)}{J}_1(\sigma')\}_D &= \{A^a, B^b\}_D T_a \otimes T_b\,, \\
\{\overset{(1)}{J}_1(\sigma), \overset{(2)}{J}_1(\sigma')\}_D &= \{B^a, B^b\}_D T_a \otimes T_b\,.
\end{aligned} \tag{4.10}$$

The calculation of $\{A^a, A^b\}_D$, $\{B^a, B^b\}_D$ and $\{A^a, B^b\}_D$ is in itself quite tedious. But it is even more challenging to express the brackets in terms of A^a and B^a and only then the algebra of the flat currents may have a closed form. This turns out to be a non-trivial problem itself. For instance, as it is seen from the above table, the S^5 coordinates do not enter covariantly in the expressions for A^6 and B^6, namely u^6 appears separately in the denominator. Nevertheless, we have calculated and found that all the brackets, $\{B^a, B^j\}_D$, $\{A^a, B^b\}_D$ and $\{A^a, A^b\}_D$, can be written as functions $F(A^a, B^a)$ of A^a and B^a.The simplest case is $\{B^a, B^b\}_D$ where the nontrivial brackets are given by[5]

$$\begin{aligned}
\{B^2, B^3\} &= \frac{e^{\phi(\sigma)}}{p^+} B^3(\sigma)\partial_\sigma\delta(\sigma-\sigma')\,, \\
\{B^2, B^4\} &= \frac{e^{\phi(\sigma)}}{p^+} B^4(\sigma)\partial_\sigma\delta(\sigma-\sigma')\,, \\
\{B^2, B^5\} &= \frac{e^{\phi(\sigma)}}{p^+} B^5(\sigma)\partial_\sigma\delta(\sigma-\sigma')\,, \\
\{B^2, B^6\} &= \frac{e^{\phi(\sigma)}}{p^+} B^6(\sigma)\partial_\sigma\delta(\sigma-\sigma')\,, \\
\{B^2, B^2\} &= \frac{e^{\phi(\sigma)}}{p^+} B^2(\sigma)\partial_\sigma\delta(\sigma-\sigma') - \frac{e^{\phi(\sigma')}}{p^+} B^2(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma')\,.
\end{aligned} \tag{4.11}$$

Using this result, one can show that $\{\overset{1}{J}_1(\sigma), \overset{2}{J}_1(\sigma')\}_D$ can be elegantly written in the following form:

$$\{\overset{1}{J}_1(\sigma), \overset{2}{J}_1(\sigma')\}_D = \left(\frac{e^{\phi(\sigma)}}{p^+} P^+ \otimes J_1(\sigma) + \frac{e^{\phi(\sigma')}}{p^+} J_1(\sigma') \otimes P^+\right)\partial_\sigma\delta(\sigma-\sigma')\,. \tag{4.12}$$

We see that the non-ultralocal terms $\sim \partial_\sigma\delta(\sigma-\sigma')$ appear in this algebra with a weight $e^{\phi(\sigma)}$. As we mentioned in the introduction, this is in parallel to the case of two-dimensional sigma models coupled to the dilaton field $\rho = e^{\phi(\sigma)}$, that are obtained from D-dimensional gravity via dimensional reduction. In this case, as was shown in [43], the presence of the dilaton field regularizes the ambiguities in the algebra of the transition matrices arising from

[5] To avoid cluttered notations, almost everywhere in brackets $\{F, G\}_D$ we omit manifest dependence on the word-sheet coordinate, and assume that the function on the left $F = F(\sigma)$, and the function on the right $G = G(\sigma')$, i.e. $\{F, G\}_D \equiv \{F(\sigma), G(\sigma')\}_D$.

the non-ultralocal terms, provided the dilaton satisfies appropriate boundary conditions. The origin of such a regularization can be traced back to the non-trivial dependence of the spectral parameter on the world-sheet coordinates and the dilaton ϕ. In the present case, the spectral parameter is a constant, but in the light-cone gauge the solution of the Virasoro constraints leads to the field $\rho \sim e^{\phi(\sigma)}$, where $\phi(\sigma)$ denotes the radial AdS_5 coordinate. It is, therefore, plausible that the non-ultralocal terms in the present case will not to lead to ambiguities in the algebra of the monodromy matrices. This issue will be analyzed in detail in a separate publication. This is in contrast with what happens if one doesn't fix the Virasoro constraints. Let us recall that in that case the algebra corresponding to (4.12) has the following form [41]:

$$\begin{aligned}\{\overset{1}{\hat{J}_1}(\sigma,t_1),\overset{2}{\hat{J}_1}(\sigma',t_2)\} = \Big(&\alpha[\Omega_P,\overset{(1)}{\hat{J}_1(\sigma,t_1)}]+\beta[\Omega_P,\overset{(2)}{\hat{J}_1(\sigma,t_2)}]+\gamma[\Omega_H,\overset{(1)}{\hat{J}_1(\sigma,t_1)}+\\ &+\overset{(2)}{\hat{J}_1(\sigma,t_2)}]\Big)\delta(\sigma-\sigma')+\Lambda\partial_\sigma\delta(\sigma-\sigma')\,,\end{aligned} \tag{4.13}$$

where in the second term $\Lambda = \Lambda(t_{1,}t_2)$ is a constant. Therefore, integrating (4.13) will inevitably lead to ambiguities [42, 54, 43].

In the appendices C and D we have collected the remaining brackets $\{A^a, B^b\}_D$, and $\{A^a, A^b\}_D$. Some of these brackets are rather complicated, however one can readily verify that all potentially dangerous terms are multiplied by $e^{\phi(\sigma)}$ and thereby likely to lead to unambiguous algebra for the monodromy matrices much along the lines in [43].

5. Algebra of currents with the spectral parameter

In this section we will collect all the pieces into the calculation of the algebra of the flat current with the spectral parameter (4.8) which leads to the monodromy matrix. Let us redefine the basis of the generators as

$$\begin{aligned}t_H^1 &= \frac{1}{2}P^- + K^-\,, \qquad t_H^2 = \frac{1}{2}P^+ + K^+\,, \qquad t_H^3 = \frac{1}{2}\bar{P}+\bar{K}\,,\\ t_H^4 &= \frac{1}{2}P+K\,, \qquad t_H^5 = 0\,, \qquad (t_H^6)_i^j = \frac{1}{2}(J_i^j+\rho^{jk}\rho_{il}J_k^l)\,.\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}t_P^1 &= \frac{1}{2}P^- - K^-\,, \qquad t_P^2 = \frac{1}{2}P^+ - K^+\,, \qquad t_P^3 = \frac{1}{2}\bar{P}-\bar{K}\,,\\ t_P^4 &= \frac{1}{2}P-K\,, \qquad t_P^5 = D\,, \qquad (t_P^6)_i^j = \frac{1}{2}(J_i^j-\rho^{jk}\rho_{il}J_k^l)\,.\end{aligned} \tag{5.2}$$

The one parameter family of flat currents, in this basis, is given by (see 2.16)

$$\hat{J}(t) = H + a(t)P + b(t)^*P\,, \tag{5.3}$$

where

$$a(t) = \frac{1+t^2}{1-t^2}\,, \qquad b(t) = \frac{2t}{1-t^2}\,, \tag{5.4}$$

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and the components of this can be written in terms of the generators as

$$\begin{aligned} H_0 &= A^a t^a_H\,, \qquad H_1 = B^a t^a_H\,, \qquad P_0 = A^a t^a_P\,, \qquad P_1 = B^a t^a_P\,,\\ {}^*P_1 &= -p^+ e^{-2\phi} P_0 = -p^+ e^{-2\phi} A^a t^a_P \equiv \bar{A}^a t^a_P\,, \end{aligned} \tag{5.5}$$

where the sum runs from a=1 to 6.

In this basis, the result of the calculation $\{\overset{1}{\hat{J}_1}(\sigma,t_1),\overset{2}{\hat{J}_1}(\sigma',t_2)\}_D$ can be written as[6]

$$\begin{aligned} \{\overset{1}{\hat{J}_1}(\sigma,t_1),\overset{2}{\hat{J}_1}(\sigma',t_2)\}_D =& -b(t_1)t^2_P\otimes[\hat{J}_0(t_2),\hat{J}_1(t_2)]e^{-\phi}\delta(\sigma-\sigma')+\\ &+b(t_2)[\hat{J}_0(t_1),\hat{J}_1(t_1)]\otimes t^2_P e^{-\phi}\delta(\sigma-\sigma')-\\ &-b(t_1)t^5_p\otimes\hat{J}_1(-t_2)\delta(\sigma-\sigma')+b(t_2)\hat{J}_1(-t_1)\otimes t^5_P\delta(\sigma-\sigma')+\\ &+t^2_H\otimes\hat{J}_1(t_2)\frac{e^{\phi}}{p^+}\partial\delta(\sigma-\sigma')+\hat{J}'_1(t_1)\otimes t^2_H\frac{e^{\phi'}}{p^+}\partial\delta(\sigma-\sigma')+\\ &+a(t_1)t^2_P\otimes\hat{J}_1(t_2)\frac{e^{\phi}}{p^+}\partial\delta(\sigma-\sigma')+a(t_2)\hat{J}'_1(t_1)\otimes\\ &\otimes t^2_P\frac{e^{\phi'}}{p^+}\partial\delta(\sigma-\sigma')-b(t_1)t^2_P\otimes\hat{J}_0(t_2)e^{-\phi}\partial\delta(\sigma-\sigma')-\\ &-b(t_2)\hat{J}'_0(t_1)\otimes t^2_P e^{-\phi'}\partial\delta(\sigma-\sigma')+\Lambda_1(t_1,t_2)e^{-\phi}e^{\phi'}\partial\delta(\sigma-\sigma')+\\ &+\Lambda_2(t_1,t_2)e^{-\phi'}e^{\phi}\partial\delta(\sigma-\sigma')+\left(S^5\right), \end{aligned} \tag{5.6}$$

where $\Lambda_{1,2}(t_1,t_2)$ are constants depending on the spectral parameters (t_1,t_2), and $\left(S^5\right)$ represents terms depending only on $e^{\phi(\sigma)}$ and coordinates u^M parameterizing S^5. The explicit form of these terms is presented in the appendix E. This algebra has a closed form structure similar to (4.13), but in this case the non-ultralocal terms are multiplied by a weight factor e^{ϕ}. In the Dirac bracket (5.6), the terms involving the AdS_5 variables have a nice algebraic structure whereas the other terms $\left(S^5\right)$ have a complicated form (see appendix E), which, however, can be simplified in some subsectors. We could not find a simpler expression for this part in general, although it may be possible to simplify this term by choosing a different parametrization or by making a gauge transformation (as, for example, is done in [18] where one makes a gauge transformation to get rid of non-local fields in the Lax pair).[7]

6. Conclusion

In this paper, we have studied the algebra of flat currents for the string on $AdS_5\times S^5$ background in the light-cone gauge with the κ symmetry fixed. We show that the currents

[6] Here we suppress the dependence on σ and use the notations $\phi'\equiv\phi(\sigma')$, $J'_1(t)\equiv J_1(\sigma',t)$.

[7] We note here that the $\left(S^5\right)$ part involves (see appendix E) the matrix U defined in (3.14) which can also be expressed in terms of components of the current $J_1(\sigma)$, namely, in terms of A^6 and B^6 given in (4.6), (4.7) respectively, as a path-ordered integral. The dependence on the weight factor e^{ϕ} arises due to the presence of such a factor in A^6. As a result, all the non-ultralocal terms are multiplied by weight factors of the form $e^{\phi(\sigma)}$.

form a closed algebra and present some explicit calculations. We point out that all the non-ultralocal terms are multiplied by weight factors $e^{\phi(\sigma)}$. From earlier results for sigma models coupled to gravity via dilaton field, this suggests that such terms are unlikely to lead to ambiguities when integrated. Further investigation of such questions including the Yangian algebra associated with the system is presently under way and will be reported later.

A. Properties of $psu(2,2|4)$

In this section we discuss some of the essential properties of the superalgebra $psu(2,2|4)$ [57–61]. Since we are interested in a supersymmetric field theory, we assume that the algebra is defined on a Grassmann space, $psu(2,2|4;\mathbb{C}B_L)$. We represent an element of this superalgebra by an even supermatrix of the form

$$G = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}, \tag{A.1}$$

where A and B are matrices with Grassmann even functions while X and Y are those with Grassmann odd functions, each representing a 4×4 matrix. (An odd supermatrix, on the other hand, has the same form, with A and B consisting of Grassmann odd functions while $X$and Y consisting of Grassmann even functions.)

An element G (see A.1) of the superalgebra $psu(2,2|4;\mathbb{C}B_L)$ is given by a 8×8 matrix, satisfying

$$GK + KG^{\ddagger} = 0\,, \tag{A.2}$$

$$\mathrm{tr}A = \mathrm{tr}B = 0\,, \tag{A.3}$$

where $K = \begin{pmatrix} \Sigma & 0 \\ 0 & I_4 \end{pmatrix}$ and $\Sigma = \sigma_3 \otimes I_2$ with I_2, I_4 representing the identity matrix in 2 and 4 dimensions respectively. The $\ddagger$ is defined by

$$G^{\ddagger} = G^{\mathrm{T}\sharp}\,, \tag{A.4}$$

where T denotes transposition and $\sharp$ is a generalization of complex conjugation which acts on the functions c of the matrices as

$$c^{\sharp} = \begin{cases} c^* & \text{(for } c \text{ Grassmann even)} \\ -ic^* & \text{(for } c \text{ Grassmann odd)} \end{cases}. \tag{A.5}$$

The condition (A.2) can be written explicitly as

$$\Sigma A^{\dagger} + A\Sigma = 0\,, \qquad B^{\dagger} + B = 0\,, \qquad X - i\Sigma Y^{\dagger} = 0\,. \tag{A.6}$$

The essential feature of the superalgebra $psu(2,2|4)$ is that it admits a $\mathbb{Z}_4$ automorphism such that the condition $\mathbb{Z}_4(H) = H$ determines the maximal subgroup to be $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ which leads to the definition of the coset for the sigma model. (This is the generalization of the $\mathbb{Z}_2$ automorphism of bosonic sigma models to the supersymmetric

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case.) The $\mathbb{Z}_4$ automorphism Ω takes an element of $psu(2,2|4)$ to another, $G \to \Omega(G)$, such that

$$\Omega(G) = \begin{pmatrix} JA^{\mathrm{T}}J & -JY^{\mathrm{T}}J \\ JX^{\mathrm{T}}J & JB^{\mathrm{T}}J \end{pmatrix}, \tag{A.7}$$

where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. It follows now that $\Omega^4(G) = G$.

Since $\Omega^4 = 1$, the eigenvalues of Ω are i^p with $p = 0, 1, 2, 3$. Therefore, we can decompose the superalgebra as

$$G = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3\,, \tag{A.8}$$

where $\mathcal{H}_p$ denotes the eigenspace of Ω such that if $H_p \in \mathcal{H}_p$, then

$$\Omega(H_p) = i^p H_p\,. \tag{A.9}$$

We have already noted that $\Omega(\mathcal{H}_0) = \mathcal{H}_0$ determines $\mathcal{H}_0 = \mathrm{SO}(4,1)\times\mathrm{SO}(5)$. $\mathcal{H}_2$ represents the remaining bosonic generators of the superalgebra while $\mathcal{H}_1, \mathcal{H}_3$ consist of the fermionic generators of the algebra. (In a bosonic sigma model, $\mathcal{H}_0, \mathcal{H}_2$ are represented respectively as Q, P.) The automorphism also implies that

$$[H_p, H_q] \in \mathcal{H}_{p+q (\text{mod } 4)}\,. \tag{A.10}$$

The space $\mathcal{H}_p$ is spanned by the generators $(t_p)_A$ of the superalgebra so that we can explicitly write

$$\begin{aligned} G &= (H_p)^A (t_p)_A \\ &= (H_0)^m (t_0)_m + (H_1)^{\alpha_1} (t_1)_{\alpha_1} + (H_2)^A (t_2)_A + (H_3)^{\alpha_2} (t_3)_{\alpha_2}\,, \end{aligned} \tag{A.11}$$

where $A = (m, \alpha_1, a, \alpha_2)$ take values over all the generators of the superalgebra, $(H_0)^m$ and $(H_2)^A$ are Grassmann even functions, while $(H_1)^{\alpha 1}$ and $(H_3)^{\alpha 2}$ are Grassmann odd functions. The generators satisfy the graded algebra $psu(2,2|4)$,

$$[(t_p)_A, (t_q)_B] = f_{AB}{}^{C} (t_{p+q})_C\,, \tag{A.12}$$

where $p + q$ on the right hand side is to be understood modulo 4.

The Killing form (or the bilinear form) $\langle H_p, H_q \rangle$ is also $\mathbb{Z}_4$ invariant so that

$$\langle \Omega\left(H_p\right), \Omega\left(H_q\right) \rangle = \langle H_p, H_q \rangle\,. \tag{A.13}$$

This implies that

$$i^{(p+q)} \langle H_p, H_q \rangle = \langle H_p, H_q \rangle\,, \tag{A.14}$$

which leads to

$$\langle H_p, H_q \rangle = 0 \qquad \text{unless} \qquad p + q = 0 \ (\text{mod} 4)\,. \tag{A.15}$$

Since the supertrace of a supermatrix M is defined as

$$\mathrm{str}(M) = \begin{cases} \mathrm{tr}A - \mathrm{tr}B \ \text{(if } M \text{ is an even supermatrix)} \\ \mathrm{tr}A + \mathrm{tr}B \ \text{(if } M \text{ is an odd supermatrix)} \end{cases}, \tag{A.16}$$

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and the metric of the algebra is defined as $G_{AB} = \mathrm{str}((t_p)_A (t_q)_B)$, the above relation also implies that only the components $G_{mn}, G_{ab}, G_{\alpha_1\alpha_2} = -G_{\alpha_2\alpha_1}$ of the metric are non-zero. The structure constants possess the graded anti-symmetry property

$$f_{AB}^{D} G_{DC} = -(-)^{|A||B|} f_{BA}^{D} G_{DC} = -(-)^{|B||C|} f_{AC}^{D} G_{DB}\,, \tag{A.17}$$

where $|A|$ denotes the Garssmann parity of A, namely, $|A|$ is 0 when A is m or a, while $|A|$ is 1 when A is α_1 or α_2.

B. Properties of $so(3,1) \oplus su(4)$

We list here some relations between the generators and the Cartan 1-forms of the $psu(2,2|4)$ algebra written in $so(3,1) \oplus su(4)$ and $so(4,1) \oplus so(5)$ basis.

$$\begin{aligned}
P^a &= \hat{P}^a + \hat{J}^{4a}\,, \qquad K^a = \frac{1}{2}\left(-\hat{P}^a + \hat{J}^{4a}\right), \qquad D = -\hat{P}^4\,, \\
J^i_j &= -\frac{i}{2}\left(\gamma^{A'}\right)^i_j P^{A'} + \frac{1}{4}\left(\gamma^{A'B'}\right)^i_j J^{A'B'}\,, \\
\hat{L}^a &= L^a_P - \frac{1}{2} L^a_K\,, \qquad \hat{L}^{4a} = L^a_P + \frac{1}{2} L^a_K\,, \qquad \hat{L}^4 = -L_D\,, \qquad \hat{L}^{ab} = L^{ab} \\
L^i_j &= \frac{i}{2}\left(\gamma^{A'}\right)^i_j L^{A'} - \frac{1}{4}\left(\gamma^{A'B'}\right)^i_j L^{A'B'}\,, \\
L^{A'} &= -\frac{i}{2}\left(\gamma^{A'}\right)^i_j L^j_i\,, \qquad L^{A'B'} = \frac{1}{2}\left(\gamma^{A'B'}\right)^i_j L^j_i\,.
\end{aligned}$$

C. $\{A^a, B^b\}_D$

We list here all the results for $\{A^a, B^b\}_D$.

1. $\{A^1, B^1\}_D = \{A^2, B^1\}_D = \cdots = \{A^6, B^1\}_D = 0.$
2. $\{A^1, B^2\}_D = \frac{e^{\phi(\sigma)}}{p^+} A^1(\sigma) B^5(\sigma) \delta(\sigma - \sigma').$
3. $\{A^2, B^2\}_D = -\frac{e^{\phi(\sigma)}}{p^+} B^2 A^5 \delta(\sigma - \sigma') - \frac{e^{\phi(\sigma')}}{p^+}\left[A^2(\sigma) + A^2(\sigma')\right] \partial_{\sigma'} \delta(\sigma - \sigma').$
4. $\{A^3, B^2\}_D = -\frac{e^{\phi(\sigma')}}{p^+} A^3(\sigma') \partial_{\sigma'} \delta(\sigma - \sigma').$
5. $\{A^4, B^2\}_D = -\frac{e^{\phi(\sigma')}}{p^+} A^4(\sigma') \partial_{\sigma'} \delta(\sigma - \sigma').$
6. $\{A^5, B^2\}_D = \frac{e^{2\phi(\sigma)}}{p^+} B^2(\sigma) \delta(\sigma - \sigma') - \frac{e^{\phi(\sigma')}}{p^+} A^5(\sigma') \partial_{\sigma'} \delta(\sigma - \sigma').$
7. $\{A^6, B^2\}_D = -\frac{e^{\phi(\sigma')}}{p^+} A^6(\sigma') \partial_{\sigma'} \delta(\sigma - \sigma').$
8. $\{A^1, B^3\}_D = 0.$
9. $\{A^2, B^3\}_D = -\frac{e^{\phi(\sigma)}}{p^+} B^3(\sigma) A^5(\sigma) \delta(\sigma - \sigma') - \frac{e^{\phi(\sigma')}}{p^+} A^3(\sigma) \partial_{\sigma'} \delta(\sigma - \sigma').$
10. $\{A^3, B^3\}_D = 0.$

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11. $\{A^4, B^3\}_D = \frac{e^{\phi(\sigma)} e^{\phi(\sigma')}}{p^+} \partial_{\sigma'} \delta(\sigma - \sigma')$.

12. $\{A^5, B^3\}_D = \frac{e^{2\phi(\sigma)}}{p^+} B^3(\sigma) \delta(\sigma - \sigma')$.

13. $\{A^6, B^3\}_D = 0$.

14. $\{A^1, B^4\}_D = 0$.

15. $\{A^2, B^4\}_D = -\frac{e^{\phi(\sigma)}}{p^+} B^4(\sigma) A^5(\sigma) \delta(\sigma - \sigma') - \frac{e^{\phi(\sigma')}}{p^+} A^4(\sigma) \partial_{\sigma'} \delta(\sigma - \sigma')$.

16. $\{A^3, B^4\}_D = \frac{e^{\phi(\sigma)} e^{\phi(\sigma')}}{p^+} \partial_{\sigma'} \delta(\sigma - \sigma')$.

17. $\{A^4, B^4\}_D = 0$.

18. $\{A^5, B^4\}_D = \frac{e^{2\phi(\sigma)}}{p^+} B^4(\sigma) \delta(\sigma - \sigma')$.

19. $\{A^6, B^4\}_D = 0$.

20. $\{A^1, B^5\}_D = 0$.

21. $\{A^2, B^5\}_D = -\frac{e^{\phi(\sigma)}}{p^+} A^5(\sigma) \partial_{\sigma'} \delta(\sigma - \sigma')$.

22. $\{A^3, B^5\}_D = 0$.

23. $\{A^4, B^5\}_D = 0$.

24. $\{A^5, B^5\}_D = \frac{e^{2\phi(\sigma)}}{p^+} \partial_{\sigma'} \delta(\sigma - \sigma')$.

25. $\{A^6, B^5\}_D = 0$.

26. $\{A^1, B^6\}_D = 0$.

27. $\{A^2, B^6\}_D = \frac{e^{\phi(\sigma)}}{p^+} \left(B^6 A^6 - A^6 B^6\right) \delta(\sigma - \sigma') + \frac{e^{\phi(\sigma)}}{p^+} A^6(\sigma) \partial_\sigma \delta(\sigma - \sigma')$.

28. $\{A^3, B^6\}_D = \{A^4, B^6\}_D = \{A^5, B^6\}_D = 0$.

29. $\{\left(A^6\right)^i_j, \left(B^6\right)^k_m\}_D = \Omega^{ik}_{jm}(\sigma) \partial_{\sigma'} \delta(\sigma - \sigma') - \left(\Omega^{ik}_{jn} \left(B^6\right)^n_m - \left(B^6\right)^k_n \Omega^{\text{in}}_{jm}\right) \delta(\sigma - \sigma')$, where

$$\Omega^{ik}_{jm} = -\frac{4e^{2\phi(\sigma)}}{p^+} \left[\frac{\left(\gamma^A U^{-1}\right)^i_j}{Tr(U)} \frac{\left(\gamma^A U^{-1}\right)^k_m}{Tr(U)} \right]. \tag{C.1}$$

We note that we have used

$$\begin{aligned}
B^6 &= (\partial_\sigma U) U^{-1}, \qquad A^6 = (\partial_\tau U) U^{-1}, \\
U &\equiv \exp\left(\frac{i}{2}\gamma^A y^A\right) = \cos\left(\left|\frac{y}{2}\right|\right) + i\gamma^A n^A \sin\left(\left|\frac{y}{2}\right|\right), \\
U^{-1} &= \exp\left(-\frac{i}{2}\gamma^A y^A\right) = \cos\left(\left|\frac{y}{2}\right|\right) - i\gamma^A n^A \sin\left(\left|\frac{y}{2}\right|\right), \\
n^A &= \frac{y^A}{|y|}, \qquad |y| = \sqrt{y^A y^A}, \qquad \left(n^A\right)^2 = 1, \\
\gamma^A &= i\rho^A \overline{\rho}^6, \qquad A = 1 \dots 5; \quad i, j = 1 \dots 4.
\end{aligned} \tag{C.2}$$

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D. $\{A^a, A^b\}_D$

We list here all the results for $\{A^a, A^b\}_D$.

1. $\{A^1, A^1\}_D = 0.$
2. $\{A^2, A^1\}_D = -\frac{e^{\phi(\sigma)}}{p^+} A^1(\sigma) A^5(\sigma)\delta(\sigma - \sigma').$
3. $\{A^3, A^1\}_D = 0.$
4. $\{A^4, A^1\}_D = 0.$
5. $\{A^5, A^1\}_D = \frac{e^{2\phi(\sigma)}}{p^+} A^1(\sigma)\delta(\sigma - \sigma').$
6. $\{A^6, A^1\}_D = 0.$
7. $\{A^2, A^2\}_D = -\frac{e^{5\phi(\sigma)}}{(p^+)^3} B^2(\sigma)\partial_{\sigma'}\delta(\sigma - \sigma') - \frac{e^{5\phi(\sigma')}}{(p^+)^3} B^2(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma').$
8. $\{A^3, A^2\}_D = -\frac{e^{5\phi(\sigma')}}{(p^+)^3} B^3(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') + \frac{e^{\phi(\sigma)}}{(p^+)} A^3 A^5 \delta(\sigma - \sigma') - \frac{e^{5\phi(\sigma)}}{(p^+)^3} B^3 B^5 \delta(\sigma - \sigma').$
9. $\{A^4, A^2\}_D = -\frac{e^{5\phi(\sigma')}}{(p^+)^3} B^4(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') + \frac{e^{\phi(\sigma)}}{(p^+)} A^4 A^5 \delta(\sigma - \sigma') - \frac{e^{5\phi(\sigma)}}{(p^+)^3} B^4 B^5 \delta(\sigma - \sigma').$
10. $\{A^5, A^2\}_D = 3\frac{e^{2\phi(\sigma)}}{p^+} A^2 \delta(\sigma - \sigma') + 2\frac{e^{\phi(\sigma)}}{p^+} A^3 A^4 \delta(\sigma - \sigma') - 2\frac{e^{5\phi(\sigma)}}{(p^+)^3} B^3 B^4 \delta(\sigma - \sigma') +$ $2\frac{e^{\phi(\sigma)}}{p^+} \left(A^5\right)^2 \delta(\sigma - \sigma') - \frac{e^{5\phi(\sigma')}}{(p^+)^3} B^5(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') - 2\frac{e^{5\phi(\sigma)}}{(p^+)^3} \left(B^5\right)^2 \delta(\sigma - \sigma').$
11. $\{A^6, A^2\}_D = -\frac{e^{6\phi(\sigma')}}{(p^+)^3} B^6(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') + 2\frac{e^{2\phi(\sigma)}}{(p^+)} \left(A^5 A^6 - B^5 B^6\right)\delta(\sigma - \sigma') +$
$\frac{e^{2\phi(\sigma)}}{2(p^+)} \left[\frac{A^6 U^{-1} A^6 U^{-1} + A^6 U^{-2} A^6 + \left(A^6\right)^2 U^2 + U A^6 U^{-1} A^6 U^2}{(1+u^6)}\right] \delta(\sigma - \sigma') -$
$\frac{e^{2\phi(\sigma)}}{2(p^+)} \left[\frac{B^6 U^{-1} B^6 U^{-1} + B^6 U^{-2} B^6 + \left(B^6\right)^2 U^2 + U B^6 U^{-1} B^6 U^2}{(1+u^6)}\right] \delta(\sigma - \sigma').$
12. $\{A^3, A^3\}_D = 0.$
13. $\{A^4, A^3\}_D = 0.$
14. $\{A^5, A^3\}_D = \frac{e^{2\phi(\sigma)}}{p^+} A^3(\sigma)\delta(\sigma - \sigma').$
15. $\{A^6, A^3\}_D = 0.$
16. $\{A^4, A^4\}_D = 0.$
17. $\{A^5, A^4\}_D = \frac{e^{2\phi(\sigma)}}{p^+} A^4(\sigma)\delta(\sigma - \sigma').$
18. $\{A^6, A^4\}_D = 0.$
19. $\{A^5, A^5\}_D = 0.$
20. $\{A^6, A^5\}_D = -2\frac{e^{2\phi(\sigma)}}{p^+} A^6(\sigma)\delta(\sigma - \sigma').$

21. $\{(A^6)^i_j,(A^6)^k_m\}_D = \frac{e^{2\phi(\sigma)}}{2p^+}\Bigg[-\frac{\left[(A^6U+UA^6)U\gamma^A\right]^i_j\left[\gamma^A U^{-1}\right]^k_m}{\sqrt{2}(1+u^6)^{3/2}}+\frac{\left[(A^6U+UA^6)U\gamma^A\right]^k_m\left[\gamma^A U^{-1}\right]^i_j}{\sqrt{2}(1+u^6)^{3/2}}+$
$\frac{(A^6)^i_j(U^{-1})^k_m}{\sqrt{2}\sqrt{(1+u^6)}}-\frac{(A^6)^k_m(U^{-1})^i_j}{\sqrt{2}\sqrt{(1+u^6)}}\Bigg]\delta(\sigma-\sigma').$

In relations 11 and 21 one should replace $(1+u^6)$ using one of the following expressions:

$$\sqrt{(1+u^6)} = \frac{\sqrt{2}}{4}\mathrm{Tr}(U)\,, \tag{D.1}$$

or

$$1+u^6 = \frac{1}{2}\left(U+U^{-1}\right)^2. \tag{D.2}$$

E. Expression for (S^5)

$$\begin{aligned}
\Lambda_1(t_1,t_2) &= b(t_1)(t^1_P\otimes t^2_H+t^2_P\otimes t^1_H+t^3_P\otimes t^4_H+t^4_P\otimes t^3_H)+\\
&+b(t_1)a(t_2)(t^1_P\otimes t^2_P+t^2_P\otimes t^1_P+t^3_P\otimes t^4_P+t^4_P\otimes t^3_P+t^5_P\otimes t^5_P)\\
\Lambda_2(t_1,t_2) &= b(t_2)(t^1_H\otimes t^2_P+t^2_H\otimes t^1_P+t^3_H\otimes t^4_P+t^4_H\otimes t^3_P)+\\
&+b(t_2)a(t_1)(t^1_P\otimes t^2_P+t^2_P\otimes t^1_P+t^3_P\otimes t^4_P+t^4_P\otimes t^3_P+t^5_P\otimes t^5_P)\\
(S^5) &= b(t_1)t^5_p\otimes\left(B^6t^6_H+a(t_2)B^6t^6_P-b(t_2)\bar{A}^6t^6_P\right)\delta(\sigma-\sigma')-\\
&-b(t_2)\left(B^6t^6_H+a(t_1)B^6t^6_P-b(t_1)\bar{A}^6t^6_P\right)\otimes t^5_P\delta(\sigma-\sigma')+\\
&+b(t_1)t^2_P\otimes(\bar{A}^6{}^k_m(B^6)^j_n\rho^{mn}\rho_{ki}(t^6_H)^i_j)\frac{e^\phi}{p^+}\delta(\sigma-\sigma')-\\
&-b(t_2)(\bar{A}^6{}^k_m(B^6)^j_n\rho^{mn}\rho_{ki}(t^6_H)^i_j)\otimes t^2_P\frac{e^\phi}{p^+}\delta(\sigma-\sigma')-\\
&-\frac{1}{2}b(t_1)a(t_2)t^2_P\otimes\Big[\bar{A}^6{}^k_m(B^6)^j_n\rho^{mn}\rho_{ki}-(B^6)^k_m\bar{A}^6{}^j_n\rho^{mn}\rho_{ki}+\\
&\qquad+\bar{A}^6{}^m_i(B^6)^n_k\rho_{mn}\rho^{kj}-(B^6)^m_i\bar{A}^6{}^n_k\rho_{mn}\rho^{kj}\Big]\times\\
&\times(t^6_P)^i_j\frac{e^\phi}{p^+}\delta(\sigma-\sigma')+\frac{1}{2}b(t_2)a(t_1)\Big[\bar{A}^6{}^k_m(B^6)^j_n\rho^{mn}\rho_{ki}-(B^6)^k_m\bar{A}^6{}^j_n\rho^{mn}\rho_{ki}+\\
&\qquad+\bar{A}^6{}^m_i(B^6)^n_k\rho_{mn}\rho^{kj}-(B^6)^m_i\bar{A}^6{}^n_k\rho_{mn}\rho^{kj}\Big]\times\\
&\times(t^6_P)^i_j\otimes t^2_P\frac{e^\phi}{p^+}\delta(\sigma-\sigma')+\\
&+b(t_1)b(t_2)\Bigg[\frac{1}{1+u^6}\Big(\bar{A}^6U^{-1}\bar{A}^6U^{-1}+\bar{A}^6U^{-2}\bar{A}^6+(\bar{A}^6)^2U^2+\\
&\qquad+U\bar{A}^6U^{-1}\bar{A}^6U^2-B^6U^{-1}B^6U^{-1}-B^6U^{-2}B^6-\\
&\qquad-(B^6)^2U^2-UB^6U^{-1}B^6U^2\Big)^i_j(t^6_P)^j_i\otimes,t^2_P\Bigg]\frac{e^\phi}{2p^+}\delta(\sigma-\sigma')+\\
&+b(t_1)\{(\bar{A}^6)^i_j,(B'^6)^k_l\}(t^6_P)^j_i\otimes(t^6_H)^l_k+b(t_2)\{(B^6)^k_l,(\bar{A'}^6)^i_j\}(t^6_H)^l_k\otimes(t^6_P)^j_i+\\
&+b(t_1)a(t_2)\{(\bar{A}^6)^i_j,(B'^6)^k_l\}(t^6_P)^j_i\otimes(t^6_P)^l_k+b(t_2)a(t_1)\{(B^6)^k_l,(\bar{A'}^6)^i_j\}(t^6_P)^l_k\otimes\\
&\otimes(t^6_P)^j_i+b(t_1)b(t_2)\{(\bar{A}^6)^i_j,(\bar{A'}^6)^k_l\}(t^6_P)^j_i\otimes(t^6_P)^l_k\,.
\end{aligned}$$

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